From comparison of the calculated values of α according to the different formulas (see Fig. 2), one can see that the spread in values is smaller and better describes the experimental data for the cube—cube elementary cell. The value of α can be calculated approximately for a structure with isolated inclusions using (55) and (59) and taking the average $\alpha = 1/2\cdot(\alpha' + \alpha'')$.

NOTATION

ε_{ij}, deformation tensor; σ_{ij}, stress tensor; C_{ijkl}, elastic modulus tensor; S_{ijkl}, compliance tensor; α_{kl}, thermal expansion tensor; T, temperature; V, volume; $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, radius vector; x_1, x_2, x_3, coordinates; E, Young's modulus; μ, shear modulus; ν, Poisson coefficient; $m_i = V_i/V$, volume concentration of the i-th component; $S_i(x_k)$, cross-sectional area of the sample occupied by the i-th component and perpendicular to the x_k axis; $\bar{S}_i(x_k) = S_i(x_k)/S(x_k)$; $S(x_k) = S_1(x_k) + S_2(x_k)$; $L_i(x_i, x_j)$, length occupied by the i-th component, perpendicular to the x_i and x_j axes; $\bar{L}_i(x_i, x_j) = L_i(x_i, x_j)/L(x_i, x_j)$; $L(x_i, x_j) = L_1(x_i, x_j) + L_2(x_i, x_j)$.

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TEMPERATURE CONDITIONS OF THE INTERACTION OF A MEDIUM WITH A THIN INCLUSION

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A mathematical model of a thin linear inclusion (layer) with a heat-liberating or thermally insulated surface is proposed for the calculation of the temperature in arbitrary bodies.

The influence of thin linear inclusions on the thermophysical state of a plane medium was studied in [1-5] using conditions of idealized thermal contact, modeling a thin intermediate layer of constant width. In [6], a different approach to such problems was proposed, consisting in modeling the inclusion in a piecewise-homogeneous plane by lines of temperature discontinuity. The temperature and temperature fluxes at an arbitrary point of the medium are completely determined [6] by the discontinuity functions. The formulation of these expressions in the condition of interaction of the medium with the inclusion, relating the values of the temperature and heat flux at opposite boundaries of the inclusion, gives a singular integro-differential equation for the desired temperature-discontinuity function [6-8]. On the basis of [9-10], it may be asserted that it is more accurate to model the inclusion by means of two discontinuities, in the temperature and in the heat flux, since in this case heat transfer

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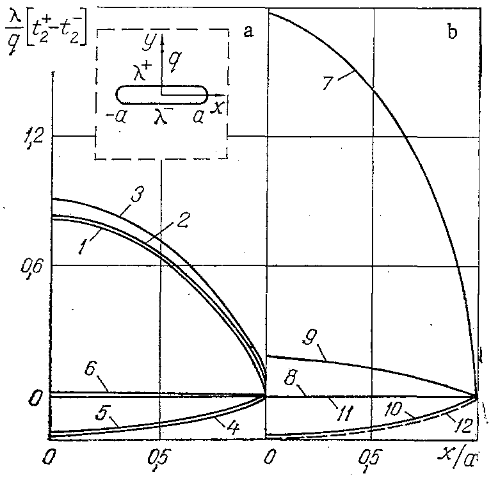

Fig. 1. Variation in the dimensionless quantity $\lambda[t_2^+(x) - t_2^-(x)]/q$ along the inclusion: a) when $\lambda_0/\lambda = 0.1$ according to models II (1), IV (2), and VI (3), and when $\lambda_0/\lambda = 10.0$ according to II (4), IV (5), and VI (6); b) when $\lambda_0/\lambda = 0$ according to II, IV, VI (7), when $\lambda_0/\lambda = 1$ according to II, IV (8), and VI (9), and when $\lambda_0/\lambda = \infty$ according to II, IV (10), and VI (11); 12) variation in the dimensionless quantity $\lambda[t_1(x, -h) - t_1(x, h)]/q$ along the abscissa.

along the axis of the inclusion is taken into account. This approach requires the use of two independent conditions of interaction, which would more completely express the physical properties of the inclusion. Formulating the results of [9], taking account of the analogy of [10], in such conditions leads to a system of integral equations determining the discontinuity functions, and consequently completely solving the given temperature problem. These interaction conditions in fact constitute the mathematical model of the inclusion: the use of particular interaction conditions allows inclusions of corresponding thermophysical type to be investigated. Thus, within the framework of the approach of [6, 9], the most important problem is the construction of relatively simple but sufficiently complete interaction conditions, allowing particular properties of the real layers to be taken into account.

Consider an inclusion of small width $2h(x)$ and thickness δ, symmetric with respect to its median line, which lies along the line $L = [-a, a]$ (Fig. 1). Suppose that the temperature $t(x, y)$ of the matrix (an arbitrary medium) is written in the form

$$t(x, y) = t_1(x, y) + t_2(x, y), \tag{1}$$

where $t_1(x, y)$ is the basic temperature field corresponding to the heat-conduction problem in the absence of inclusions and heat sources on the abscissa and $t_2(x, y)$ is the perturbation of the temperature field due to the presence of inclusions and heat sources.

Quantities pertaining to the inclusion will be denoted by the subscript 0. The boundary values of the functions when the argument tends to L from above and below will be denoted by the superscripts + and −, respectively.

Between the inclusion and the matrix, over the whole thickness, there is ideal thermal contact

$$\begin{gathered} t_0(x, \pm h) = t(x, \pm h), \\ (x \in L), \\ \lambda_0 \frac{\partial t_0(x, \pm h)}{\partial y} = \lambda^{\pm} \frac{\partial t(x, \pm h)}{\partial y}. \end{gathered} \tag{2}$$

Using the hypothesis of a small inclusion width, it may be written that

$$\frac{\partial t_0(x, \pm h)}{\partial y} \cong \pm \frac{t_0(x, \pm h) - t_0(x, 0)}{h}, \quad t(x, \pm h) \cong t_1(x, \pm h) + t_2^{\pm}(x). \tag{3}$$

To simplify the mathematical analysis of the problem (with no significant loss in generality), the contact line of the inclusion with the matrix is taken to be L

$$\frac{\partial t(x, \pm h)}{\partial y} \cong \frac{\partial t^{\pm}(x)}{\partial y}. \tag{4}$$

Using Eqs. (2) and (3), the first condition of interaction of the medium with a thin linear inclusion is obtained, taking account of its geometry:

$$\lambda^{+} \frac{\partial t_2^{+}(x)}{\partial y} + \lambda^{-} \frac{\partial t_2^{-}(x)}{\partial y} - \lambda_0 [t_2^{+}(x) - t_2^{-}(x)]/h(x) = F(x), \tag{5}$$

where

$$F(x) = -2\lambda^{\pm} \frac{\partial t_1^{\pm}(x)}{\partial y} + \lambda_0 [t_1(x, h) - t_1(x, -h)]/h(x). \tag{6}$$

Taking into account that $t_1(x, \pm h) \cong t_1^{\pm}(x) \pm h(x) \frac{\partial t_1^{\pm}(x)}{\partial y}$, it follows that

$$F(x) = -2\lambda^{\pm} \left[1 - \frac{\lambda_0 (\lambda^{+} + \lambda^{-})}{2\lambda^{+}\lambda^{-}} \right] \frac{\partial t_1^{\pm}(x)}{\partial y}. \tag{7}$$

In some investigations, in particular [11], the model adopted rests on the hypothesis that the heat flux through the inclusion is proportional to the temperature difference at its boundaries, taking no account of its real width, i.e., all the conditions of thermal contact have been taken at the axis of the inclusion, which leads to simpler but less accurate relations: in the interaction conditions in Eq. (5), the right-hand side then takes the form [11]

$$F(x) = -2\lambda^{\pm} \frac{\partial t_1^{\pm}(x)}{\partial y}. \tag{8}$$

The heat balance equation for the elementary volume takes the form

$$\frac{dq_x}{dx} + \frac{dq_y}{dy} + \frac{q_n}{\delta} + q_0(x, y) = 0, \tag{9}$$

where q_x, q_y, q_n are the heat fluxes in the directions x, y, n; $q_0(x, y)$ is the density of the heat sources.

Suppose that the heat transfer at the side surfaces of the inclusion satisfies the law $q_n = f(x, y, t_0, \alpha, \beta, \ldots)$. Then, using Fourier's law [12], it follows that

$$\Delta t_0(x, y) - \frac{f(x, y, t_0, \alpha, \beta, \ldots)}{\delta\lambda_0} + \frac{q_0(x, y)}{\lambda_0} = 0. \tag{10}$$

In the particular case when $f(x, y, t_0, t_e, \alpha_0) = \alpha_0(t_0 - t_e)$ (Newtonian heat transfer), Eq. (10) coincides with the heat-conduction equation for a plate with a temperature field that is symmetric with respect to its median plane.

Averaging Eq. (10) with respect to y and integrating with respect to x, it is found that

$$2\lambda_0 h(x) \left[\frac{\partial t_0^c(x)}{\partial x} - \frac{\partial t_0^c(-a)}{\partial x} \right] + \lambda_0 \int_{-a}^{x} \left[\frac{\partial t_0(\xi, h)}{\partial y} - \frac{\partial t_0(\xi, -h)}{\partial y} \right] d\xi -$$

$$-2\int_{-a}^{x} h(\xi)[f^c(\xi)/\delta - q_0^c(\xi)]\, d\xi = 0, \tag{11}$$

where

$$\left\{\frac{\partial t_0^c(\xi)}{\partial x},\ f^c(\xi),\ q_0^c(\xi)\right\} = \frac{1}{2h(x)}\int_{-h}^{h}\left\{\frac{\partial t_0(\xi, y)}{\partial x},\ f(\xi, y, t_0, \alpha, \beta, \ldots),\ q_0(\xi, y)\right\} dy.$$

Taking account of Eq. (2), the assumption

$$\frac{\partial t(x, \pm h)}{\partial x} \cong \frac{\partial t^{\pm}(x)}{\partial x},\quad \frac{\partial t_0^c(x)}{\partial x} \cong \frac{1}{2}\left[\frac{\partial t_0(x, h)}{\partial x} + \frac{\partial t_0(x, -h)}{\partial x}\right],$$

and Eq. (4), the second condition of interaction between the medium and a thin linear heat-extracting inclusion is obtained, taking account both of the heat loss from the side surface of the inclusion and the heat transfer along its axis

$$\lambda_0 h(x)\left[\frac{\partial t^+(x)}{\partial x} + \frac{\partial t^-(x)}{\partial x} - 2\frac{\partial t_0^c(-a)}{\partial x}\right] +$$

$$+\int_{-a}^{x}\left[\lambda^+ \frac{\partial t_2^+(\xi)}{\partial y} - \lambda^- \frac{\partial t_2^-(\xi)}{\partial y}\right] d\xi - 2\int_{-a}^{x} h(\xi)\,[f^c(\xi)/\delta - q_0^c(\xi)]\, d\xi = 0\quad (x \in L). \tag{12}$$

The quantity $[\partial t_0^c(-a)]/(\partial x)$ may be determined from certain *a priori* assumptions. In particular, it may be assumed that

$$\frac{\partial t_0^c(-a)}{\partial x} = \min\left\{\left[\frac{\lambda_0}{\max(\lambda, \lambda_0)}\right]^{\varphi},\ \left[\frac{\lambda}{\max(\lambda, \lambda_0)}\right]^{\psi}\right\}\frac{\partial t_1(-a)}{\partial x}\quad (\varphi, \psi > 0), \tag{13}$$

which ensures accurate satisfaction of the interaction conditions in the readily analyzable limiting cases.

When x = a, Eq. (12) takes the form

$$\int_L\left[\lambda^+ \frac{\partial t_2^+(x)}{\partial y} - \lambda^- \frac{\partial t_2^-(x)}{\partial y}\right] dx = 2\int_L h(x)\,[f^c(x)/\delta - q_0^c(x)]\, dx \equiv Q \tag{14}$$

and determines the amount of heat Q liberated through the side surface of the inclusion in unit time on account of heat transfer.

A simpler version of the second interaction condition may be proposed. Suppose that the inclusion is thermally orthotropic ($\lambda_{0x} = 0$), i.e., there is no heat propagation in the longitudinal direction, which may be the case in the presence of insulating barriers in the inclusion, e.g., in the transverse motion of a liquid heat carrier inside a set of practically heat-impervious channels. In fact $\frac{dq_x}{dx} = -\lambda_{0x}\frac{\partial^2 t_0}{\partial x^2} = 0$ in this case, and Eq. (10) takes the form

$$\frac{\partial^2 t_0(x, y)}{\partial y^2} - f(x, y, t_0, \alpha, \beta, \ldots)/(\delta\lambda_0) + q_0(x, y)/\lambda_0 = 0. \tag{15}$$

Averaging over y results in a simplified version of the second interaction condition for the medium and a thin heat-extracting inclusion

$$\left[\lambda^+ \frac{\partial t_2^+(x)}{\partial y} - \lambda^- \frac{\partial t_2^-(x)}{\partial y}\right] - 2h(x)\,[f^c(x)/\delta - q_0^c(x)] = 0. \tag{16}$$

The conditions obtained in Eqs. (5) and (12) or (16) form six basic complete mathematical models of a thin heat-extracting inclusion: I) Eqs. (5), (6), (12); II) Eqs. (5), (6), (16);

III) Eqs. (5), (7), (12); IV) Eqs. (5), (7), (16); V) Eqs. (5), (8), (12); VI) Eqs. (5), (8), (16). Model IV was tested in [6-8], and model VI, in the case with no heat transfer, in [11].

In particular, it is interesting to consider the possible limiting cases of the equations of model I, which give the most complete and accurate description of an inclusion of homogeneous isotropic material:

a) $\lambda_0 = 0$ (thermally insulated inclusion)

$$\lambda^+ \frac{\partial t^+(x)}{\partial y} + \lambda^- \frac{\partial t^-(x)}{\partial y} = 0, \tag{17}$$

$$\lambda^+ \frac{\partial t_2^+(x)}{\partial y} - \lambda^- \frac{\partial t_2^-(x)}{\partial y} = 0. \tag{18}$$

It is taken into account here that the presence of heat sources and heat transfer from the surface of such a layer is physically meaningless.

b) $\lambda_0 = \infty$ (absolutely heat-conducting inclusion)

$$t_2^+(x) - t_2^-(x) + t_1(x, h) - t_1(x, -h) = 0, \tag{19}$$

$$\frac{\partial t^+(x)}{\partial x} + \frac{\partial t^-(x)}{\partial x} - 2\frac{\partial t_0^c(-a)}{\partial x} = 0. \tag{20}$$

It may readily be noted from Eq. (19) that $t(x, h) - t(x, -h) = 0$, i.e., the temperature inside the inclusion is unchanged over the width. Then Eq. (20), when $\partial t_0^c(-a)/\partial x = 0$, ensures constancy of the temperature along the axis of the inclusion. The latter requirement must be taken into account in *a priori* relations of the type in Eq. (13).

c) $\lambda_0 = \lambda^+ = \lambda^- = \lambda$ (thermophysical equivalence of the materials of the inclusion and the matrix)

$$t_2^+(x) - t_2^-(x) - h(x)\left[\frac{\partial t_2^+(x)}{\partial y} + \frac{\partial t_2^-(x)}{\partial y}\right] = 2h(x)\frac{\partial t_1^\pm(x)}{\partial y} - [t_1(x, h) - t_1(x, -h)]$$

or if $t_1(x, \pm h)$ is expanded in Taylor series on the right-hand side

$$t_2(x, h) - t_2(x, -h) = O(h^2) \cong 0, \tag{21}$$

i.e., in fact there is no perturbation of the temperature field

$$h(x)\left[\frac{\partial t^+(x)}{\partial x} + \frac{\partial t^-(x)}{\partial x} - 2\frac{\partial t_0^c(-a)}{\partial x}\right] + \int_{-a}^{x}\left\{\frac{\partial t_2^+(\xi)}{\partial y} - \frac{\partial t_2^-(\xi)}{\partial y} - \frac{2h(\xi)}{\lambda}\left[\frac{f^c(\xi)}{\delta} - q_0^c(\xi)\right]\right\} d\xi = 0. \tag{22}$$

It is quickly evident that in this case it is required that $\partial t_0^c(-a)/\partial x = \partial t_1(-a)/\partial x$, which must also be taken into account in *a priori* relations of the type in Eq. (13). The other models, apart from III, do not permit accurate realization of the limiting cases so completely.

The given method permits the generalization of the interaction conditions obtained for the medium and the thin linear heat-extracting inclusion to the case of a curvilinear longitudinal axis of the inclusion and also to the nonsteady case.

As an example, consider the plane steady heat-conduction problem neglecting heat transfer through the side surfaces for a homogeneous ($\lambda^+ = \lambda^- = \lambda$) plate with a thin inclusion of elliptical form $h(x) = 0.1\sqrt{a^2 - x^2}$, taking account of the action of a heat source of power q at the point $(0, a)$. Using the given method [6, 9], the heat-conduction problem for plane media with thin linear inclusions is solved by comparing models II, IV, and VI from the viewpoint of their applicability for the investigation of inclusions with different thermophysical properties. Analogously to [6, 9], systems of integrodifferential equations that may be solved by the method of orthogonal Chebyshev polynomials with an accuracy of up to 1% may be constructed. Since the heat transfer through the side surface of the inclusion is neglected, the discontinuity of the heat fluxes of the perturbed temperature field, within the framework of the given models, is zero. The results of comparing the values of the discontinuities in the perturbed temperature field obtained using models II, IV, and VI as a function of the dimensionless parameter λ_0/λ are shown in Fig. 1. Models II and IV are very satisfactory in the limiting cases $\lambda_0 = 0$, $\lambda_0 = \lambda$, and $\lambda_0 = \infty$; model II provides relatively more accurate results as $\lambda_0 \to \infty$ and model IV when $\lambda_0 \cong \lambda$. The difference in the results when models II and IV are used is no more than 1-3%. Use of model VI is expedient only in the case $\lambda_0 \cong 0$, since when $\lambda_0 > 0.5\lambda$ the error in comparison with the more accurate models II and IV exceeds 100%. Note that, when $\lambda_0 \cong 0$, the choice of model is unimportant, since all the models give practically identical results. With variation in the distance of the heat source from the inclusion, the qualitative picture of the comparison of the models remains the same.

NOTATION

x, y, Cartesian coordinates; $h(x)$, halfwidth of inclusion; δ, thickness of inclusion; a, halflength of inclusion; $L = [-a, a]$, median line of inclusion; $t_0(x, y)$, $t(x, y)$, temperature inside and outside inclusion; λ_0, λ, thermal conductivity of inclusion and matrix; λ^+, λ^-, $t_j^{\pm}(x)$, limiting values of λ and the functions $t_j(x)$ ($j = 1, 2$) at the abscissa when $y > 0$ and $y < 0$; n, external normal to the plane of the inclusion; q_x, q_y, q_n, heat fluxes in the directions x, y, n; $q_0(x, y)$, density of heat sources in the inclusion; α, β, ..., parameters of the external medium; Q, rate of heat transfer through the side surface of the inclusion; α_0, heat-transfer coefficient; $t_e(x, y)$, temperature of the external medium; λ_{0x}, thermal conductivity in the direction x of a thermophysically orthotropic inclusion; q, power of source; φ, ψ, arbitrary exponents in the *a priori* relations.

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